Agency Problems, Recapitalization Costs and Resolution of Financial Distress

2 authors, including:

Santiago Moreno-Bromberg
University of Zurich

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Santiago Moreno–Bromberg† Quynh-Anh Vo‡

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Abstract

We consider, within a dynamic–contracting framework with moral hazard, the possibility of recapitalization as an alternative to liquidation when a firm is in financial distress. In fact, we show that firm recapitalization may arise in an optimal, long–term contract. As a consequence, we find that there are two mechanisms at a firm's disposal so as to deal with financial difficulties: one corresponds to a recapitalization process, the other to a liquidation one. The choice of mechanism is based on a cost–benefit analysis.

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†E-mail: santiago.moreno@bf.uzh.ch. Address: University of Zürich, Department of Banking and Finance, Plattenstrasse 32, 8032 Zürich.

‡E-mail: quynhanh.vo@bf.uzh.ch. Address: University of Zürich, Department of Banking and Finance, Plattenstrasse 32, 8032 Zürich.

1 Introduction

There are two basic scenarios that a firm which is unable to honor its commitments towards its investors may face: either it is liquidated by the investors and ceases to operate or it is recapitalized by its owners and continues operations as an ongoing concern. Our aim in this work is to study how should the choice between these two options be made. More specifically, we examine the implications of agency conflicts and external financing costs on the firm's decisions regarding how to deal with financial distress, as well as on the firm's value. In our model, the mechanism for addressing financial distress functions as a device that provides good–management incentives to corporate insiders.

We start with a standard dynamic, moral–hazard setting where an entrepreneur raises funds from outside investors to finance an investment project. The contractual relationship is hindered by the agency problem arising from the fact that the entrepreneur is better informed about her project and actions than the investors are. The only way for the latter to provide incentives to the former lies in the right to terminate the relationship and seize all assets, should the entrepreneur not be able to make the promised repayments. Hence, in this model, when facing a business failure, the firm has only one option, namely to let the investors trigger a liquidation procedure.

In practice however, a distressed firm can also execute an equity infusion to avoid default. Distressed equity issuances are not rare. Jostarndt (2009) reports that in Germany, between 1996 and 2004, 123 out of 267 financially–troubled corporations issued new equity. Franks and Sanzhar (2006) document that distressed equity issuances were a significant proportion of total seasoned issuances in the United Kingdom from 1989 to 1998. The present paper fills this gap by introducing into the afore-described framework the possibility of recapitalization.

We analyze a scenario in which a risk–neutral entrepreneur contracts with risk–neutral investors to finance a business project. The project, once funded and running, produces at each date an observable binary cash flow, whose distribution depends on an unobservable effort exerted by the entrepreneur. We assume, for simplicity, that the set of feasible effort levels consists of two elements: *high* and *low*. The distribution of the cash flows under high effort first–order stochastically dominates the one under low effort. Exerting low effort, however, provides the entrepreneur with private benefits. To induce the entrepreneur to choose the high effort level, the investors can use performance–based incentives, where the entrepreneur is rewarded in the form of bonus payments in case of good performance, while she has to bear some punishment if an unsatisfactory performance is realized. In line with the literature, we allow the investors to punish the entrepreneur by terminating the contract, which results in the liquidation of the firm. In addition, we consider the possibility that the investors may impose some pecuniary penalty on the entrepreneur. In that case, there will be a monetary transfer from the latter to the former. Such transfer, which represents a new capital injection by the entrepreneur, can be interpreted as the recapitalization of the firm, given that, in our setup, the entrepreneur is not only the manager of the firm but also the firm's shareholder.

Both liquidation and recapitalization are costly in our model. The cost of liquidation is due to the fact that the value of the firm as a gone concern is less than the firm's value as an ongoing one. The cost of recapitalization is motivated by the observation that, in general, issuing new equity is a costly process. Moreover, the firm's recapitalization in the present framework is voluntary in the sense that it is in the entrepreneur's interest to perform it. We thus assume that the maximum amount the entrepreneur is willing to inject is determined by the expected, discounted value of the cash flows accruing to her if the firm continues to operate.

We explicitly characterize, in an infinite horizon setting, the optimal contract between the investors and the entrepreneur. This has interesting implications for the firm's dynamics. In our characterization, the expected, discounted utility of the entrepreneur acts as a state variable summarizing all past performances of the firm. We first show that the total firm value is increasing and concave with respect to the entrepreneur's promised utility. This obeys the fact that the incentives problem is relaxed when the entrepreneur's stake in the firm's cash flows increases. We also find that, for incentives purposes, the total utility of the entrepreneur is sensitive to the cash flows: it increases following a high cash flow and decreases following a low one. The degree of the said sensitivity is increasing in the magnitude of the moral–hazard problem.

According to the optimal contract, the firm is in financial difficulties following a business failure whenever the entrepreneur's promised utility is low. We find that the optimal mechanism to deal with financial distress has a dichotomic structure that depends on whether recapitalization is more or less costly than liquidation. In the first case, once the firm has fallen into financial distress, it should be liquidated and no recapitalization will be employed. In the second scenario, the firm will be recapitalized up to the extent that the liquidation risk is totally eliminated. Furthermore, by considering an implementation of the optimal contract via debt, equity and cash reserves, we find that any distressed equity issuance is accompanied by debt concession, which is in line with the stylized fact reported by Franks and Sanzhar (2006).

The analysis of the optimal contract also allows us to derive several comparative–statics results. For instance, we find that the net issuance proceeds are decreasing in the recapitalization costs and increasing in the volatility of cash flows. To complement our analytical findings, we conduct a numerical analysis in which we find that: 1) the firm's value is decreasing with the recapitalization costs; 2) the marginal value of cash increases with recapitalization costs and the volatility of cash flows; 3) when the moral–hazard problem becomes more severe, it is more likely that the liquidation regime is the optimal financial–distress mechanism; 4) the recapitalization regime is more likely to be the optimal one when the liquidation value of the firm is low.

Our work is closely related to the dynamic–agency models of DeMarzo and Fishman (2007b), Biais et al. (2004) and Biais et al. (2007). These authors analyze how financial contracts can be designed so as to mitigate the agency conflicts between investors and entrepreneurs. Nevertheless, they focus on the use of liquidation for incentive purposes, and do not allow for recapitalization possibilities. Our model extends the setup in Biais et al. (2004) by introducing the option of recapitalization, which we show may arise as an alternative to liquidation.

Our assumption that some new capital can be injected into the firm during its lifespan is similar to Clementi and Hopenhayn (2006) and DeMarzo and Fishman (2007a), who use models of multi–period borrowing/lending under asymmetric information to explain some facts regarding the firm's investment decisions, growth and survival rate. However, while in those papers the new capital contribution results in a growth of the firm, in the present model the additional capital serves to repay debt, so that the firm can be maintained as an ongoing concern.

The optimal dynamic contracting problem is also analyzed in continuous–time settings by, among others, DeMarzo and Sannikov (2006), Sannikov (2008), Biais et al. (2010) and Hoffmann and Pfeil (2010). Although a discrete–time setting is more intuitive, the main advantage of a continuous–time approach is its tractability, which stems from the differential equations that characterize the optimal contract. We opt for a discrete–time approach, our setup being sufficiently tractable to allow us to fully solve for the optimal contract.

The organization of the paper is as follows: In Section 2 we describe the model. In Section 3 we analyse the main properties of the optimal contract. Section 4 discusses further implications of our optimal contract. Finally, we conclude in Section 5. All proofs are provided in the Appendix.

2 The Model

2.1 Environment

We work in a discrete–time, infinite–horizon setting. Our economy consists of an entrepreneur and a group of investors. All agents are risk neutral and discount the future at the same rate r. The entrepreneur has access to a risky project that requires a start–up capital of I, which exceeds her initial wealth A. Hence, she needs to raise funds from investors. Once the the latter have agreed to provide financing, a firm will be created to operate the project[1].

The cash flows generated by the project in period t are represented by the random variable R_t, which, for simplicity, we assume takes only two values: Whenever the project is successful, the entrepreneur collects a high cash flow R_+. If the project fails, the cash flow is R_-, which is strictly smaller than R_+. The project's probability of success in any period t depends exclusively on the current effort e_t exerted by the entrepreneur. This results in cash flows that are independent across periods. Again, for the sake of simplicity, we assume that only two effort levels are possible: $e_t = 1$ (*high effort*) and $e_t = 0$ (*low effort*). The probability of success corresponding to the effort level e_t is denoted by $p(e_t)$ where:

$$p(e_t) = \begin{cases} p & \text{if} \quad e_t = 1; \\ p - \Delta p & \text{if} \quad e_t = 0; \end{cases}$$

and $0 < \Delta p < p < 1$ are given. Furthermore, depending on the level of effort exerted, the entrepreneur enjoys a private benefit equal to $B\,(1 - e_t)$, where B is strictly positive. Hence, hard work by the entrepreneur improves the expected profitability of the project, but it also prevents her from enjoying private benefits. The entrepreneur's effort is unobservable to outsiders and can, therefore, not be contracted upon.

At the beginning of each period, and as an incentives–providing device, the investors may decide to liquidate the firm. We shall denote by x_t the firm's *continuation probability*, so $1 - x_t$ is the probability that the firm is liquidated. In the event of liquidation, the non–negative proceeds from selling the firm's assets are L. Notice that, as long as the project is a constant returns to scale technology, x_t may also be interpreted as the size of the firm at the beginning of period t. In other words, the investors may decide to downsize a firm that is not functioning in a satisfactory fashion. Both interpretations are frequently found in the literature.

If the firm is not liquidated in period t, the project generates the cash flow R_t. After the cash–flow realization is observed, some monetary transfer, denoted by c_t, to or from the entrepreneur may occur. In the present setting, we do not make the standard assumption that c_t must always be non–negative[2]. In contrast, we allow c_t to be positive or negative. A positive c_t is a cash payment to the entrepreneur, which serves as a reward to the latter. A negative c_t corresponds to a new capital injection by the entrepreneur into the firm. This negative payment plays the role of another punishment device besides liquidation. Given that the entrepreneur is also the firm's inside shareholder, it allows us to consider the possibility of recapitalization.

An explanation regarding what we mean by *recapitalization* is worth providing here. In

[1] In the present setting, the entrepreneur is not only the manager of the firm but also a shareholder. She holds a fraction of the equity, and the remainder is own by outside investors. This is not an assumption but, in fact, a feature of the firm's capital structure that implements the financing contract between the entrepreneur and investors (see Section 4).

[2] This assumption is usually referred to in the literature as a *limited–liability constraint*, which, although appropriate in a static setting, is too strong in a dynamic one. Indeed, in a dynamic framework the firm is treated as ongoing concern; thus, the entrepreneur can accept to inject capital if it allows the firm to continue and generate income for her in the future. We impose below a weaker type of limited–liability constraint, which insures that the entrepreneur's total utility (not current income) is non negative.

practice, the term recapitalization is commonly used to indicate any transaction that leads to a significant change in a firm's capital structure[3]. Possible transactions include, for example, debt–for–equity swaps, equity–for–debt swaps and issuances of fresh equity. In this work, what we mean by recapitalization is an infusion of fresh equity by the firm's shareholders. Furthermore, our formulation of recapitalization corresponds to the case where equity issuances are used to maintain the firm as an ongoing concern. They occur when the firm is financially distressed. The issuance's proceeds are not used to finance new investment opportunities, but rather to reinforce the firm's balance sheet in order to cope with commitments towards creditors. Consequently, in our model recapitalization does not lead to any change in the firm's size, which contrasts with the investment modeling proposed by Clementi and Hopenhayn (2006) and DeMarzo and Fishman (2007a). In these papers, there can also be some new capital injected into the firm in each period. Such capital contribution, however, is interpreted as investment, since it results in an expansion of the firm.

It should be noted that in many jurisdictions, when a firm makes an equity issuance, the firm's existing shareholders have the right, but not the obligation, to acquire new shares in proportion to the shares they already held. Only shares that are not taken up by existing shareholders are offered to the public. From their sample of distressed firms that issued equity in the UK during the 1989 - 1998 period, Franks and Sanzhar (2006) document that take–up by insiders is, on average, higher than by outsiders (79% vs. 52.8%), which is a signal of the management's confidence in the restructuring. Our formulation of recapitalization does not impose any restriction on whether or not existing and outside shareholders have to purchase new shares. We require, however, that inside shareholders take up their entitlement of the new issuance.

We capture the following two features of recapitalization: First, in practice, the process of issuing new equity to recapitalize a firm typically involves substantial costs. Ross et al. (2008) estimate total direct costs of seasoned equity offerings by US corporations from 1990 to 2003 to be, on average, 6.72% of the gross proceeds. Franks and Sanzhar (2006) report, when studying an UK sample of distressed equity issuers from 1989 to 1998, that the costs of underwriting new equity are 12.74% of the market value of the existing equity. To model this feature, we assume that the entrepreneur has to bear a cost τ for each unit of capital injected. More precisely, we postulate that the utility function of the entrepreneur takes the following form:

$$U(c_t) = \begin{cases} c_t & if \quad c_t \geq 0; \\ (1+\tau)\, c_t & if \quad c_t < 0. \end{cases}$$

Therefore, injecting $|c_t|$ into the firm implies total costs of $(1+\tau)|c_t|$ to the entrepreneur. Another interpretation is that for each monetary unit of new equity issued, the firm only receives $1/(1+\tau)$ in cash. Hence, $\tau/(1+\tau)$ represents the marginal cost of recapitalization[4]. Second, the firm's shareholders should agree to contribute new funds only if the gains from preserving the firm as an ongoing concern at least outweigh the cost they must incur. To formalize this, we impose the following condition, which bounds the amount the entrepreneur is willing to inject into the firm from below:

$$U(c_t) + \frac{w_{t+1}}{1+r} \geq 0 \quad \text{for all } t \in \mathbb{N}, \tag{1}$$

where w_t represents the expected continuation utility of the entrepreneur at the beginning of period t. Since w_t is nonnegative[5], Condition (1) is trivially satisfied if c_t is positive. In the

[3]See, for instance, Berk and DeMarzo (2010).

[4]In fact, recapitalization costs contain some elements that are fixed and others that are proportional. For simplicity, we only consider proportional costs.

[5]Otherwise, the entrepreneur should terminate the contractual relationship and quit.

case where c_t is negative, we have that the present value to the entrepreneur of the firm kept as an ongoing concern should be at least sufficient to cover the total recapitalization costs. Condition (1) can also be seen as a limited–liability constraint of sorts, and it is clearly weaker than the standard constraints $c_t \geq 0$ and $w_{t+1} \geq 0$ that are commonly imposed in the literature.

We define the expected cash flows under high and low effort as

$$\overline{R} := pR_+ + (1-p)R_- \quad \text{and} \quad \underline{R} := (p - \Delta p)R_+ + (1 - p + \Delta p)R_-,$$

respectively. We make the following assumptions on the parameters of the model:

Assumption 1. *The project is profitable only if the entrepreneur monitors it carefully, i.e.*

$$\frac{1+r}{r}\overline{R} > I > \frac{1+r}{r}(\underline{R} + B).$$

Assumption 2. *Early liquidation is inefficient, i.e.* $L < \frac{1+r}{r}\overline{R}$.

The timeline within each period is summarized in Figure 1: first the firm can be continued or liquidated; if the firm is continued, the entrepreneur decides on her effort; then a cash flow is realized; finally a monetary transfer to or from the entrepreneur may occur.

2.2 Formulating the Optimal Contracting Problem

The financing contract between the entrepreneur and the investors is designed and agreed upon in period 0. For the time being, we assume that all parties can commit to a long-term contract $\Phi = \{x_t, c_t\}_{t\in\mathbb{N}}$ specifying, based on the entire history of observed cash flows, continuation probabilities and monetary transfers to/from the entrepreneur. The entrepreneur then chooses an effort strategy $e = \{e_t\}_{t\in\mathbb{N}}$ in order to maximize her expected utility. A strategy e is said to be incentive compatible if it maximizes the entrepreneur's expected utility given the contract $\Phi = \{x_t, c_t\}_{t\in\mathbb{N}}$. Therefore, an incentive compatible contract is described by a triple $\{e^*, x, c\}$, where e^* is an incentive compatible strategy that the investors want to induce. We focus our attention on the case where it is optimal to provide incentives to the entrepreneur so that she exerts high effort in all periods, i.e. e_t^* equals 1 for all t in $\mathbb{N}$[6].

The contractual problem can be formulated recursively using the expected continuation utility of the entrepreneur w_t as the state variable. This is a well–known fact in repeated moral hazard models (see, e.g. Spear and Srivastava (1987)). Along the equilibrium path, where high effort is exerted, w_t is given by:

$$w_t = \mathbb{E}\left[\sum_{s=t}^{\infty} \frac{\left(\prod_{u=t}^{s} x_u\right) U(c_s)}{(1+r)^{s-t}} \Big| \mathcal{F}_t\right],$$

where the filtration $\mathcal{F} = \{\mathcal{F}_t\}_{t\in\mathbb{N}}$ summarizes the problem's information structure. In each period t, for a given w_t the optimal contract specifies: a continuation probability x_t; conditional on the firm not being liquidated, monetary transfers to/from the entrepreneur, $\overline{c}_t$ and $\underline{c}_t$; continuation utilities for the entrepreneur, $\overline{w}_{t+1}$ and $\underline{w}_{t+1}$, contingent on whether in period t the cash–flow realization is R_+ or R_- (see Figure 1).

Let us define the alternative state variable w_t^c via

$$w_t^c = \frac{w_t}{x_t} \quad \text{for all } t \in \mathbb{N}. \tag{2}$$

[6] In fact, to ensure that the optimal level of effort is always the high effort, additional conditions are necessary. For further discussion regarding this point we refer the reader to Biais et al. (2004).

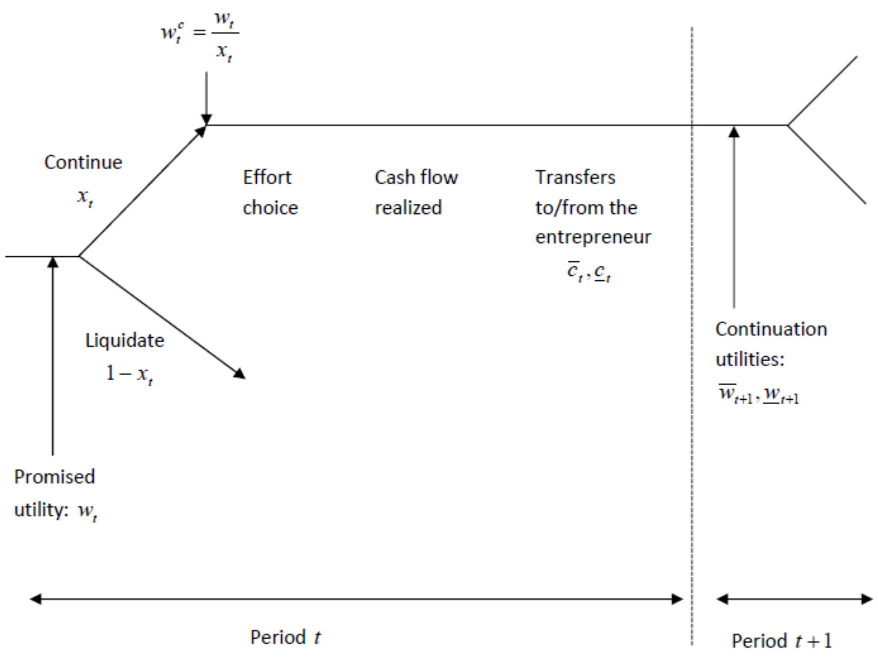


Figure 1: The timeline.

While w_t represents the expected continuation utility of the entrepreneur at the beginning of period t and prior to the liquidation decision, w_t^c stands for the entrepreneur's expected continuation utility conditional on the firm being continued in that period. Since we are working in an infinite horizon setting with a constant discount factor and cash flows are independent and identically distributed (i.i.d.), the optimal contract is stationary. We may then drop all the time subscripts.

Let us denote by $F(w)$ the highest possible continuation utility for the investors, given a continuation utility w promised to the entrepreneur. The function F satisfies the following Bellman equation:

$$F(w) = \max_{x,\overline{c},\underline{c},\overline{w},\underline{w}} x\left[\overline{R} - p\overline{c} - (1-p)\underline{c} + \frac{pF(\overline{w}) + (1-p)F(\underline{w})}{1+r}\right] + (1-x)L$$

for all w greater than or equal to 0, subject to the following constraints:

I. **Promise keeping.** The amount that the entrepreneur expects to receive at the beginning of each period must be equal to the sum of the utility she derives from the transfer paid to/from her during that period plus the expected present value of her continuation utility:

$$w = x\left[pU(\overline{c}) + (1-p)U(\underline{c}) + \frac{p\overline{w} + (1-p)\underline{w}}{1+r}\right]. \tag{3}$$

II. **Incentives to exert high effort in every period.** Since the cash flows are i.i.d., the following temporary incentive compatibility constraint (ICC) is sufficient to make the contract $\sigma = \{x_t, c_t\}_{t \in \mathbb{N}}$ incentive compatible:

$$x\left[pU(\overline{c}) + (1-p)U(\underline{c}) + \frac{p\overline{w} + (1-p)\underline{w}}{1+r}\right] \geq x\left[(p - \Delta p)\, U(\overline{c}) + (1 - p + \Delta p)U(\underline{c}) + \frac{(p-\Delta p)\,\overline{w} + (1-p+\Delta p)\underline{w}}{1+r} + B\right].$$

After simplification this yields

$$U(\overline{c}) - U(\underline{c}) + \frac{\overline{w} - \underline{w}}{1+r} \geq \frac{B}{\Delta p}. \tag{4}$$

Notice that the provision of incentives is spread between the current transfers and the continuation utilities. This illustrates the benefit of an enduring relationship between the entrepreneur and the investors: it allows the latter to smooth out the cost of incentive compatibility over time.

III. **Recapitalization constraint.** The entrepreneur is only willing to recapitalize the firm if the following condition is fulfilled:

$$U(c) + \frac{w}{1+r} \geq 0 \text{ for both pairs } (\overline{c}, \overline{w}) \text{ and } (\underline{c}, \underline{w}).$$

As a consequence of Constraint (4), the above constraint can be reduced to

$$U(\underline{c}) + \frac{\underline{w}}{1+r} \geq 0. \tag{5}$$

IV. **Feasibility.** The continuation utilities and the continuation probability must satisfy:

$$(\overline{w}, \underline{w}) \in \mathbb{R}_+^2 \quad \text{and} \quad x \in [0, 1]. \tag{6}$$

3 Properties of the Optimal Contract

3.1 The Firm's Value Function

We start by analyzing some properties of the *firm's value function* which is denoted as V: for a given continuation utility for the entrepreneur w, we have $V(w) = F(w) + w$. The value $V(w)$ is given by

$$V(w) = \max_{x,\overline{c},\underline{c},\overline{w},\underline{w}} \left\{ x\left[\overline{R} + p\big(U(\overline{c}) - \overline{c}\big) + (1-p)\big(U(\underline{c}) - \underline{c}\big) + \frac{pV(\overline{w}) + (1-p)V(\underline{w})}{1+r}\right] + (1-x)L \right\} \tag{7}$$

for all w greater than or equal to 0, subject to the Constraints (3) - (6). The following proposition summarizes some key properties of the firm's value function:

Proposition 1. *The value function V is concave and strictly increasing on $[0, \frac{1+r}{r}\frac{pB}{\Delta p})$. On $[\frac{1+r}{r}\frac{pB}{\Delta p}, +\infty)$ the value function satisfies $V \equiv \frac{1+r}{r}\overline{R}$.*

To develop some intuitive understanding, we should keep in mind that the higher the entrepreneur's stakes in the firm's future cash flows are, the less severe the moral–hazard problem becomes. When the utility level promised to the entrepreneur decreases, the moral hazard problem gains relevance, which in turn leads to a higher risk of liquidation or recapitalization. Both liquidation and recapitalization are socially costly. Liquidation is so since, even at the time of liquidation, the project remains potentially profitable. Recapitalization is also costly because of the costs borne by the firm's owners. Therefore, the firm's value is non–decreasing with respect to the entrepreneur's rent w. When the entrepreneur's rent reaches $\frac{1+r}{r}\frac{pB}{\Delta p}$, the asymmetric–information problem vanishes. Accordingly, the firm's value reaches its first-best level which equals the present value of a perpetual annuity of $\overline{R}$.

Concerning the concavity of V, this property implies that the marginal contribution of a small increase in the entrepreneur's rent to enhancing the firm's value becomes smaller when this rent is already at a high level. This is due to the fact that when w is small, liquidation and recapitalization risks are very high. Consequently, a small augmentation of w will have a significant impact on the firm's value by virtue of reducing these risks. In contrast, when w is already high, these risks are small and further reducing them does not significantly affect the firm's value.

3.2 Liquidation, Recapitalization and Payments to the Entrepreneur

We now turn our attention to the detailed provisions of the optimal contract. It follows directly from Proposition 1 that on $[\frac{1+r}{r}\frac{pB}{\Delta p}, +\infty)$, the optimal contract is given by

$$x = 1, \overline{w} = \underline{w} = \frac{1+r}{r}\frac{pB}{\Delta p}, \underline{c} = w - \frac{1+r}{r}\frac{pB}{\Delta p} \text{ and } \overline{c} = w - \frac{1+r}{r}\frac{pB}{\Delta p} + \frac{B}{\Delta p}.$$

In particular, notice that in this region the moral–hazard friction vanishes. As a consequence, the entrepreneur's continuation utility is independent of the outcomes: the firm will be continued forever.

In order to determine the optimal contract on $\big[0, \frac{1+r}{r}\frac{pB}{\Delta p}\big)$, we proceed in three steps. We first address, in Lemma 1, how the choice between current transfers and continuation utilities as instruments to remunerate the entrepreneur is made.

Lemma 1. *Whenever w is smaller than $\frac{1+r}{r}\frac{pB}{\Delta p}$, the payments $\overline{c}$ and $\underline{c}$ made to the entrepreneur are non–positive.*

In other words, as long as c is positive, decreasing it in exchange for an increase of the entrepreneur's continuation utilities results in a higher value of the firm. Hence, given that both parties are risk neutral and discount the future at the same rate, it is optimal to postpone current compensations to the entrepreneur, in favor of her continuation utilities, until the moral hazard problem becomes irrelevant.

Due to Lemma 1 and the fact that $U(c)$ equals $(1+\tau)c$ for non–positive values of c, we can rewrite the promise–keeping Constraint (3), for w on $\big[0, \frac{1+r}{r}\frac{pB}{\Delta p}\big)$, as follows:

$$p\overline{c} + (1-p)\underline{c} = \frac{1}{1+\tau}\Big(\frac{w}{x} - \frac{p\overline{w} + (1-p)\underline{w}}{1+r}\Big). \tag{8}$$

Inserting Equation (8) into the Objective Function (7), we find that, on $[0, \frac{1+r}{r}\frac{pB}{\Delta p})$, the Bellman equation determining to the firm's value may be rewritten as follows [7]:

$$V(w) - \frac{\tau}{1+\tau}w = \max_{x,\underline{w},\overline{w}} \Big\{x\Big[\overline{R} + \frac{p}{1+r}\big(V(\overline{w}) - \frac{\tau}{1+\tau}\overline{w}\big) + \frac{1-p}{1+r}\big(V(\underline{w}) - \frac{\tau}{1+\tau}\underline{w}\big)\Big] + (1-x)L\Big\} \tag{9}$$

[7] Note that $p\big(U(\overline{c}) - \overline{c}\big) + (1-p)\big(U(\underline{c}) - \underline{c}\big) = \tau\big(p\overline{c} + (1-p)\underline{c}\big)$ whenever $\overline{c}, \underline{c} \leq 0$.

subject to the Constraints (3) - (6) together with

$$\overline{c}, \underline{c} \leq 0. \tag{10}$$

In terms of the liquidation policy, we find that it is optimal to choose a positive probability to liquidate the firm if and only if the entrepreneur is promised a utility that is strictly smaller than $pB/\Delta p$.

Lemma 2. *Given an expected utility w promised to the entrepreneur, it is optimal to set the firm's continuation probability to*

$$x = \min\left\{1, w\frac{\Delta p}{pB}\right\}.$$

Lemma 2 allows us to express the alternative state variable w^c defined in Expression (2), and which we replace for w in the sequel, in the following way:

$$w^c = \begin{cases} w & if \quad w \geq \frac{pB}{\Delta p} \\ \\ \frac{pB}{\Delta p} & if \quad 0 \leq w < \frac{pB}{\Delta p} \end{cases} \tag{11}$$

We have presented in Lemmas 1 and 2 some crucial features of the optimal termination probability and current transfers. This sets us up to examine the (optimal) continuation utilities. Since the incentive compatibility constraint is binding on $[0, \frac{1+r}{r}\frac{pB}{\Delta p})$, we may, on this domain, express $\left(\underline{c}, \overline{c}\right)$ as follows:

$$\overline{c} = \frac{1}{1+\tau}\left(w^c + \frac{(1-p)B}{\Delta p} - \frac{\overline{w}}{1+r}\right) \quad \text{and} \quad \underline{c} = \frac{1}{1+\tau}\left(w^c - \frac{pB}{\Delta p} - \frac{\underline{w}}{1+r}\right). \tag{12}$$

Combining Constraints (5), (6) and (10) with Expression (12) yields the following constraints for $(\underline{w}, \overline{w})$:

$$\frac{\overline{w}}{1+r} \geq w^c + \frac{(1-p)B}{\Delta p} \quad \text{and} \quad \frac{\underline{w}}{1+r} \geq w^c - \frac{pB}{\Delta p}. \tag{13}$$

We observe from the Objective Function (9) that two cases must be distinguished: either recapitalization costs are high, corresponding to the case

$$V'(0) < \frac{\tau}{1+\tau}$$

or they are low, corresponding to the case

$$V'(0) > \frac{\tau}{1+\tau}.$$

We present in Proposition 2 the characterization of the optimal contract. In order to do so, we define the following thresholds:

$$w^* := \frac{1+r}{r}\frac{pB}{\Delta p}, \quad w^{**} := \frac{pB}{\Delta p}\left(1 + \frac{1}{1+r}\right) \quad \text{and} \quad w^{***} := \frac{pB}{\Delta p}. \tag{14}$$

Proposition 2. *The optimal contract is characterized by three regimes:*

(i) When w^c belongs to $\left[w^*, \infty\right)$ the entrepreneur receives positive payments in the current period:

$$\overline{c} = w^c - w^* + \frac{B}{\Delta p} \quad \textit{and} \quad \underline{c} = w^c - w^*.$$

The entrepreneur's continuation utilities are $\overline{w} = \underline{w} = w^*$. *The firm will be operated in the next period with probability one independently of the current cash flow.*

(ii) *When* w^c *belongs to* $\left[w^{**}, w^*\right)$ *no current payment is made to the entrepreneur:* $\overline{c} = \underline{c} = 0$. *The entrepreneur is provided with continuation utilities as follows:*

$$\overline{w} = (1+r)\left(w^c + \frac{(1-p)B}{\Delta p}\right) \quad \textit{and} \quad \underline{w} = (1+r)\left(w^c - \frac{pB}{\Delta p}\right).$$

The firm will be in operation in the following period with probability one independently of the current cash flow.

(iii) *When* w^c *belongs to* $\left[w^{***}, w^{**}\right)$ *we have:*

(a) *Following a high cash–flow realization, the entrepreneur gets a zero payment:* $\overline{c} = 0$. *The firm is operated with certainty in the next period and the entrepreneur's continuation utility is*

$$\overline{w} = (1+r)\left(w^c + \frac{(1-p)B}{\Delta p}\right).$$

(b) *Following a low cash–flow realization, there are two possibilities:*

- *If the recapitalization costs are high, the firm is liquidated in the next period with probability*

$$1 - x = 1 - \frac{(1+r)\left(w^c - \frac{pB}{\Delta p}\right)}{\frac{pB}{\Delta p}}.$$

If the firm is continued, the utility $\underline{w}$ *promised to the entrepreneur is equal to* $(1+r)\left(w^c - \frac{pB}{\Delta p}\right)$.

- *If the recapitalization costs are low, the entrepreneur has to recapitalize the firm by an amount of absolute value*

$$-\underline{c} = \frac{1}{1+\tau}\left(w^{**} - w^c\right)$$

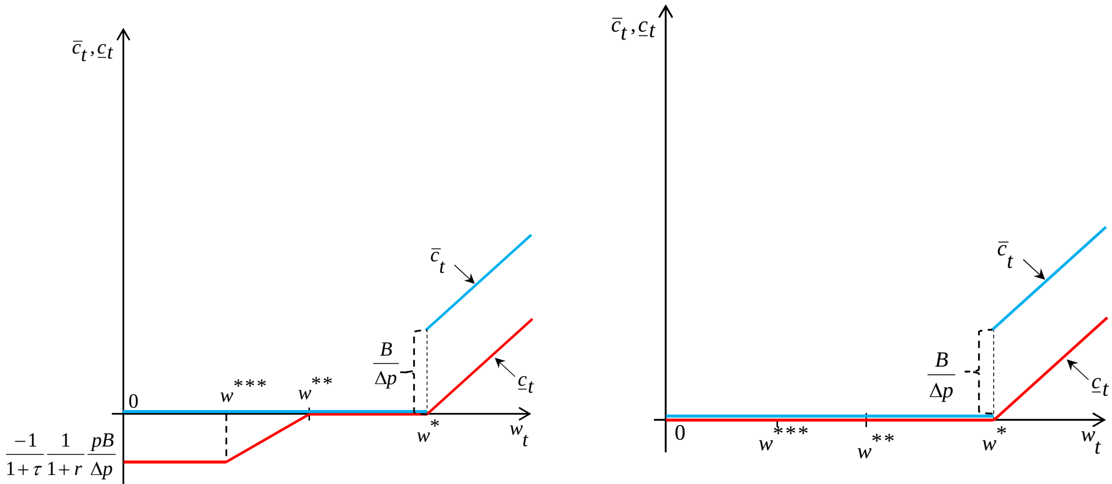
and whereby can avoid liquidation. The firm is operated in the next period with probability 1 *and the entrepreneur's continuation utility is*

$$\underline{w} = \frac{pB}{\Delta p} = w^{***}.$$

For the sake of clarity, we graphically represent in Figures 2, 3 and 4 the optimal contract described in Proposition 2.

Evolution of the entrepreneur's utility. In general, the entrepreneur's total utility moves in the same direction as the cash–flow realization: It increases following a high realization and decreases following a low one. It stops being sensitive to the firm's cash flows only when the firm's accumulated performance reaches a certain threshold (see Items (*i*)). The sensitivity of the entrepreneur's payoff to the firm's performance serves for incentive purposes. Moreover, the degree of this sensitivity depends on the magnitude of moral hazard problem. Indeed, define the measure of sensitivity k as

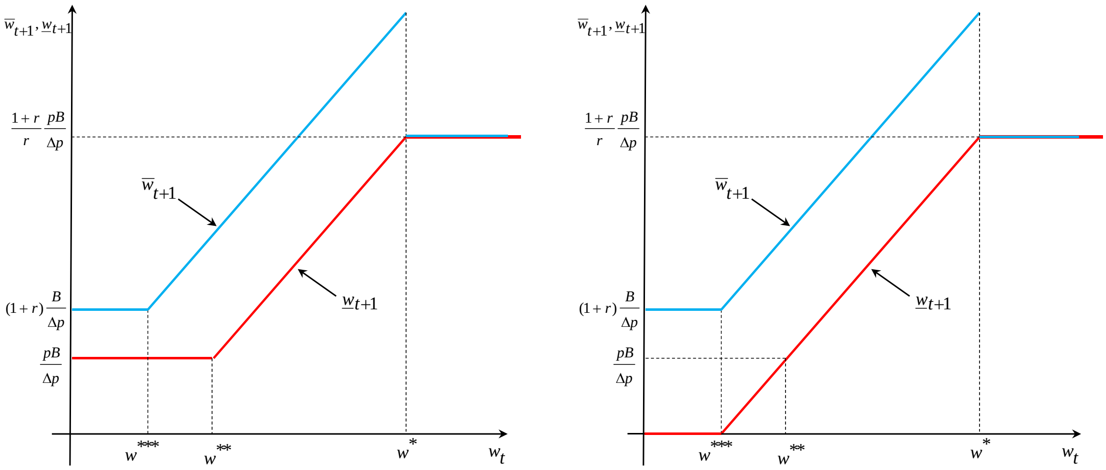
$$k := \frac{B}{\Delta p\left(R_+ - R_-\right)} < 1.$$



(a) Low recapitalization cost (b) High recapitalization cost

Figure 2: **The optimal monetary transfers**

Current transfers are positive whenever the entrepreneur's promised utility w_t reaches the threshold w^*. It happens, in case of low recapitalization costs, that the monetary transfer is negative (i.e. a recapitalization occurs) following a low cash–flow realization whenever w_t is lower than w^{**},



(a) Low recapitalization cost (b) High recapitalization cost

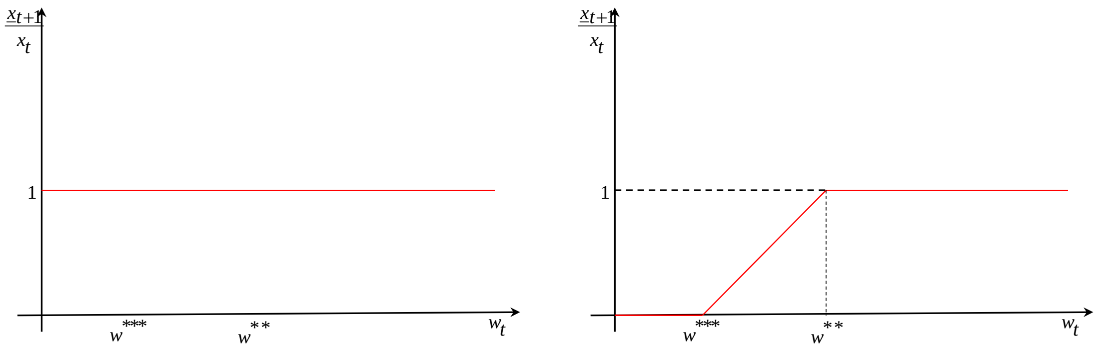
Figure 3: **The optimal continuation utilities**

We observe that $\underline{w}_{t+1} \leq w_t \leq \overline{w}_{t+1}$. In the case of low recapitalization costs, neither $\overline{w}_{t+1}$ nor $\underline{w}_{t+1}$ fall below $\frac{pB}{\Delta p}$, regardless of the value of w_t. This shows that liquidation (or downsizing) never occurs. In the case of high recapitalization costs, $\underline{w}_{t+1}$ may fall below $\frac{pB}{\Delta p}$ if w_t is less than w^{**}. Hence, in this case liquidation (or downsizing) happens after a failure if the utility promised to the entrepreneur is lower than w^{**}.

We can rewrite the continuation utilities characterized in Item (ii) as follows:

$$w_{t+1} = (1+r)\big(w_t^c + k\,(R_t - \overline{R})\big).$$

Observe that k is increasing in the magnitude of the moral–hazard problem, captured by the fraction $B/\Delta p$.



(a) Low recapitalization cost (b) High recapitalization cost

Figure 4: **The optimal liquidation policy**
The figure shows the optimal liquidation probabilities after a low cash–flow realization.

Financial distress. The firm faces financial difficulties following a business failure when the entrepreneur's stake in the firm is low. There are two mechanisms for the firm to deal with financial distress:

- In the first mechanism, the firm is liquidated with a positive probability. Interpreting x as the firm size, this means that the firm should be downsized. The fraction of the firm which is liquidated is decreasing with the entrepreneur's stakes in it[8].

- According to the alternative mechanism, the firm will be recapitalized up to the extent that the liquidation risk is eliminated. In other words, after being recapitalized, the firm will continue in operation with full certainty. The net recapitalization amount decreases with the entrepreneur's stakes in the firm and the recapitalization costs.

Examining the motion of the entrepreneur's continuation utility, we find that, according to the first mechanism, the firm never resorts to the recapitalization option when it faces financial difficulties. In contrast, following the second mechanism, once the firm is created, no liquidation or downsizing will occur during its lifetime. Each time the firm falls into financial distress, it will be recapitalized in order to completely avoid liquidation. Hence, the two above-described financial–distress mechanisms correspond, respectively, to pure liquidation and pure recapitalization procedures. In our optimal contract, we do not observe that, after a business failure, the firm is sometimes liquidated and other times recapitalized.

Which mechanism is to be implemented hinges on whether liquidation or recapitalization is costlier. Given that the the function V represents the firm's total value, the monotonicity of the mapping $w \mapsto V(w) - \frac{\tau}{1+\tau}w$ accounts for the marginal cost of liquidation relative to the marginal cost of recapitalization. Intuitively, the first mechanism is superior (inferior) to the second when the liquidation option is less costly (costlier) than the recapitalization option.

3.3 Initiating the Contract

Proposition 2 provides the characterization of the optimal contract given an initial utility w_0 promised to the entrepreneur. To complete the characterization of the dynamics of the

[8]Notice from Figures 3(b) and 4(b) that if the firm's current size is already less than 1, i.e. w_t is less than w^{***}, which implies $w_t^c = \frac{pB}{\Delta p}$, the firm will be closed after a business failure. In other words $\underline{w}_{t+1}$ equals 0, therefore $\frac{\underline{w}_{t+1}}{\frac{pB}{\Delta p}}$ is also equal to zero.

contractual relationship, we must then specify w_0, whose equilibrium value depends on the allocation of bargaining power between the entrepreneur and the investors. If the investors compete to fund the project, then w_0 is such that the entrepreneur receives the highest utility possible, provided that the investors break even:

$$w_0 = \max \ \{w \in R_+ \mid F(w) \geq I - A\}.$$

On the contrary, if the investors have all the bargaining power, w_0 is determined via:

$$w_0 = \arg\max_{w \in R_+} F(w).$$

Note that if the optimal value of w_0 is less than $pB/\Delta p$, the probability that the project will be funded is smaller than one.

4 Implications of the Optimal Contract

So far we have characterized the optimal contract in terms of an optimal mechanism. In order to gain more insight on the optimal resolution of financial distress and on the various impacts of the recapitalization costs, we consider in this section an implementation of the above–described optimal mechanism. Then, we conduct a numerical analysis to derive several comparative statics results that describe how the firm's value and the optimal financial–distress mechanism vary with the recapitalization costs, the magnitude of agency problem and the volatility of the cash flows.

4.1 Implementation

We now describe, in line with Biais et al. (2004) and Biais et al. (2007), an implementation of the optimal contract via debt, equity and cash reserves. Since our main contribution lies in the introduction of the recapitalization option, we focus on the implementation of the contract when recapitalization costs are low.

Cash reserves. The firm holds cash reserves so as to meet any necessary cash outlays. The change in the level of cash reserves in each period is equal to the net operating cash flow, plus the interest on cash reserves, minus the payments to the entrepreneur and the financiers. Since no further investments are made after date 0, these additional cash reserves are accounted for as retained earnings in the firm's financial statements.

Debt. Debt consists of securities that generate periodic payments.

Equity. Cash flows that are not used to pay debt claims may be used to pay a dividend to equity holders. Dividends are paid in proportion to share ownership.

In this implementation, pay–offs and decisions of the firm are contingent on the level of cash reserves. Put differently, the level of cash reserves plays the role of a record–keeping device, as the entrepreneur's promised utility w_t does in the abstract characterization of the optimal contract. For implementation purposes, we define three thresholds:

$$m^* := \frac{1+r}{r}p(R_+ - R_-), \quad m^{**} := (1 + \frac{1}{1+r})p(R_+ - R_-) \quad \text{and} \quad m^{***} := p(R_+ - R_-).$$

Proposition 3. *The optimal contract in the case of low recapitalization costs is implemented by a combination of debt, equity and cash reserves:*

- ***Creation of the firm***. *the firm is financed by debt and equity. The entrepreneur contributes her initial wealth A and is granted a fraction k of the equity. Investors hold the remaining fraction $1-k$ of the equity and all the debt. The firm uses contributed funds to pay the investment cost and to hoard an amount of cash $M_0 = w_0/k$:*

  - *If $M_0 \geq m^{***}$, the firm is financed at full scale, i.e. the size of the investment equals I.*
  - *If $M_0 < m^{***}$, the firm is financed at a scale smaller than 1, i.e. the size of the investment is equal to $\frac{M_0}{m^{***}} I$.*

- ***Normal functioning of the firm***. *In each period $t \in \mathbb{N}$, the firm pays a fixed amount $\overline{R}$ to its debt holders. Whenever the size–adjusted cash reserves $m_t = \frac{w_t^c}{k}$ reach m^*, the firm distributes dividends*

$$d_t = m_t - m^* + \frac{1}{k}\frac{B}{\Delta p}\mathbb{1}_{\{R_t = R_+\}}.$$

- ***Financial distress***. *When m_t falls below m^{**}, after a low cash flow the firm is restructured as follows: an amount*

$$i_t = m^{**} - m_t$$

  of new equity is issued[9] *and the current debt service is written off by τi_t. A fraction k of new issuance proceeds is taken up by the entrepreneur.*

The proposed implementation has several features consistent with stylized facts:

Firm financial structure. The firm's liability structure includes debt and equity. Debt is only held by the investors, and equity is held by both the entrepreneur and the investors. Debt corresponds to securities that generate a sequence of fixed payments. Equity is a claim on the dividends that are distributed when cash reserves reach a contractually–specified threshold. The dividend boundary m^* is increasing in the volatility of the cash flows.

Financial distress. The firm is recapitalized when the size–adjusted cash reserves are low. The net issuance proceeds $\frac{1}{1+\tau} i_t$ are decreasing in the recapitalization costs and increasing in the volatility of cash flows. Moreover, equity issuance is accompanied by debt concession. This is consistent with a stylized fact reported by Franks and Sanzhar (2006): creditors concessions are found to accompany the distressed equity issuance in 30.6% of their sample firms.

4.2 Numerical Analysis

We complement our analysis with some numerical results. Our baseline parameter values are as follows: under high effort, the project is successful with probability equal to 0.7, and its expected cash flows are $\overline{R} = 8$ per period. Exerting low effort reduces the success probability by $\Delta p = 0.2$ but provides the entrepreneur with private benefits of $B = 2$. The project has a liquidation value of $L = 86$. The riskless interest rate is 10%. For this choice of parameters, we numerically solve the constrained optimization problem that generates the value function V. This boils down to an iterative procedure to find the unique fixed point of the mapping $v \mapsto Tv$ defined via the Objective Function (7) subject to the Constraints (3) - (6)[10]. In order to distinguish the value functions arising in two possible cases, namely high or low recapitalization costs, we denote in the sequel the value function V by V_l in the first case and by V_r in the second one.

[9] Note that because of issuance costs, the net cash inflow is $\frac{1}{1+\tau} i_t$

[10] We make our code available upon request.

First, we pin down the cut–off threshold for the value of τ that determines the boundary between the two mechanisms used to deal with financial distress. We plot in Figure 5 the value of $V_r'(0)$ and $V_l'(0)$ as function of τ [11], together with the graph of the function $f(\tau) = \frac{\tau}{1+\tau}$. These graphs intersect at $\tau^* \simeq 0.19$. We observe that, when $\tau < \tau^*$, it holds that $V_r'(0) > V_l'(0) > \frac{\tau}{1+\tau}$. For values of τ above this threshold, the inequalities are reverted. Hence, if the proportional costs of recapitalization are less than 19%, the firm is optimally recapitalized when it falls into financial distress. In the other case, the firm could be liquidated should it find itself in financial difficulties.

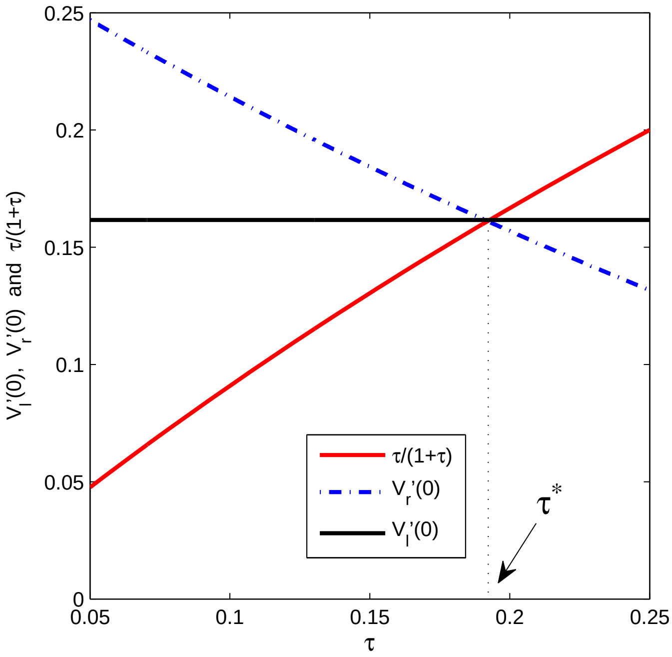


Figure 5: The cut–off threshold for τ

Figure 6 shows how the recapitalization costs affect the firm's value. We observe that as long as τ is less than τ^*, the firm's value is *strictly* decreasing with τ. When τ increases above τ^*, there is a switch from the recapitalization regime to the liquidation regime, and then firm value is independent of τ.

We plot in Figures 7(a) and 7(b) the first derivative of the value function V for different levels of the recapitalization costs and the volatility of cash flows, respectively[12,13]. It is interesting to look at this since as we have seen in the previous subsection, the first derivative of the value

[11]Since $V_l'(0)$ is independent of τ, we find in Figure 5 that the graph of $V_l'(0)$ is an horizontal line at the value 0.1616

[12]To vary the volatility, while keeping the expected cash flows fixed at 8, we set $R_+ \equiv 15$. This implies that, as a function of the probability of success, $R_-(p) = (8 - 15p)/(1 - p)$. After simplifications we obtain $\mathrm{Var}[R](p) = 49p/(1-p)$; thus, the volatility of cash flows is increasing in p. We have set $\tau = 0.15$ in our example.

[13]A disclaimer is in order here. We have approximated the value of V_r via an iterative algorithm that converges quite rapidly in the topology of uniform convergence. As finer grids are employed, the convergence to the true solution is guaranteed. The latter, however, is not the case for V_r'. This is due to the fact that the linear interpolates of the iterates are, à priori, not concave. This is akin to a *chattering solution* in the context of the Calculus of Variations (see, eg. Ekeland and Turnbull (1983)). In order to deal with this issue we compute the smallest, piecewise–linear, concave majorant of each iterate and its corresponding derivative. This results in a sequence of uniformly convergent concave functions, whose derivatives, by the Arzelà–Ascoli, also converge uniformly. The said uniform limit is a good approximation of the true derivative, but it is very

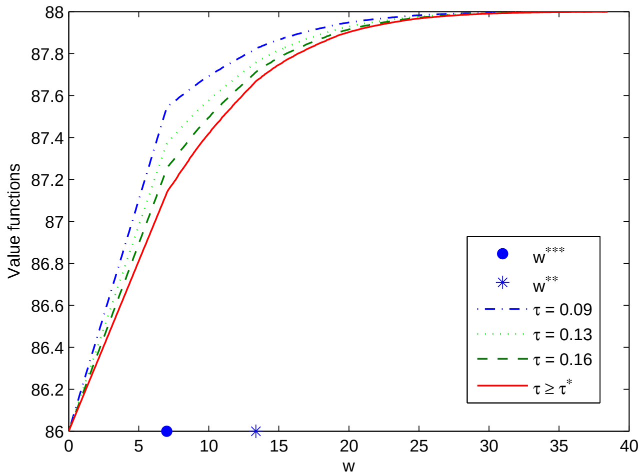
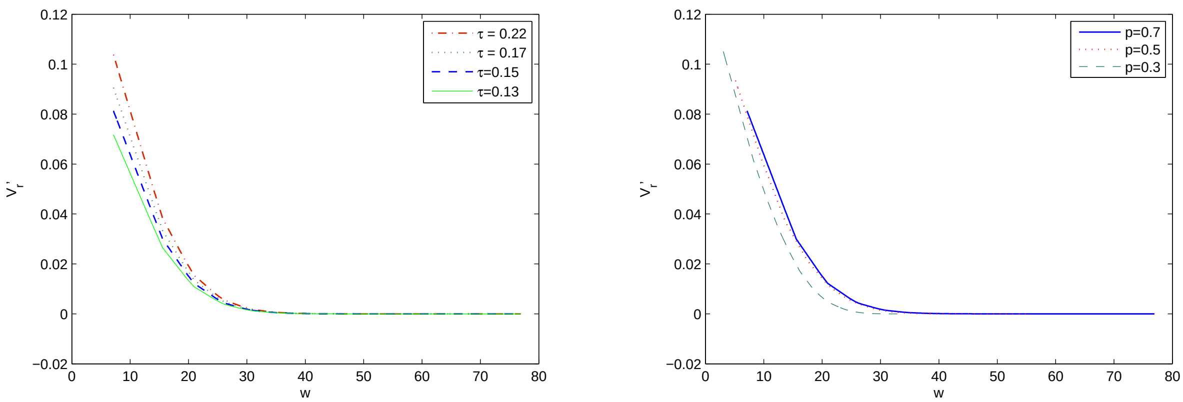


Figure 6: The firm's value for different values of the recapitalization cost τ

function V can be interpreted as the marginal value of cash. We observe that intuitively, the firm's value is more responsive to changes in their cash reserves when recapitalization costs or volatility of cash flows are high.



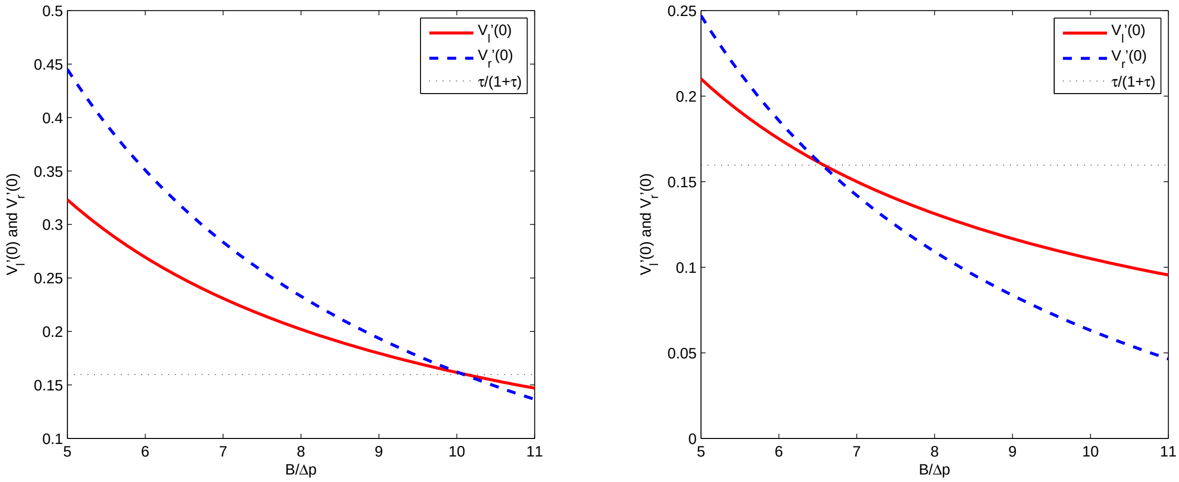
(a) Different values of the recapitalization cost τ.

(b) Different levels of the volatility of cash flows.

Figure 7: The marginal value of cash

To illustrate the effect of the magnitude of the moral–hazard problem, captured by the fraction $B/\Delta p$, as well as that of the liquidation value L, on the optimal resolution of financial distress, we plot in Figure 8 the first derivative of V_l and V_r at 0 as functions of $B/\Delta p$ for two values of L. The plots correspond to the choice $\tau = \tau^*$, the remaining parameters being as above. We observe that when the moral hazard problem becomes more severe, it is more likely that the liquidation regime is the optimal one. Concerning the liquidation value L, when it decreases (say from $L = 86.7$ in Figure 8(b) to $L = 86$ in Figure 8(a)), liquidation becomes more costly, which explains why the recapitalization regime is more likely to be the optimal one.

non–smooth. For illustration purposes, the plots in Figures 7(a) and 7(b) have been smoothed out. We refer the reader to Choné and LeMeur (2001) and Ekeland and Moreno-Bromberg (2010), as well as the references therein, for a thorough discussion on the difficulties of approximating derivatives in variational problems when convexity/concavity constraints are imposed.



(a) $L = 86$ (b) $L = 86.7$

Figure 8: The effects of $B/\Delta p$ and L on the optimal financial–distress mechanism

In order to illustrate the dynamics of the entrepreneur's continuation utility, we simulate the firm's cash–flow dynamics $R = \{R_t\}_{t\in\mathbb{N}}$, which follow at each date a binomial distribution with $p = 0.7$. We take $R_+ = 15$ and $R_- = -25/3$ so that the expected cash flows per period is exactly 8. We assume that the entrepreneur has no initial capital ($A = 0$), the initial investment costs I are equal to 70 and the investors are competitive. Hence, the initial utility w_0 promised to the entrepreneur is determined by:

$$w_0 = \max \{w \in R_+ \mid F(w) \geq I - A\}.$$

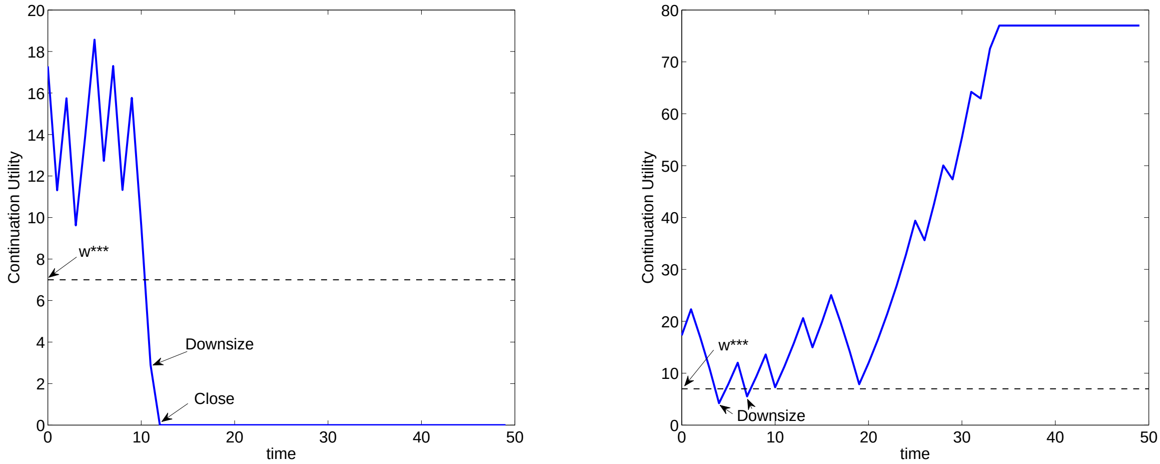
We plot in Figure 9 two paths of the entrepreneur's continuation utility when recapitalization costs are high. In the first path, the firm is closed after two successive failures. In the second path, the firm is downsized two times but it recovers after each failure and after approximately 32 periods the entrepreneur's continuation utility reaches the absorbing threshold $w^* = 77$.

Figure 10 shows a path of the entrepreneur's continuation utility in the case of low recapitalization costs[14]. We see that it never falls below $pB/\Delta p$, which means that the firm is never downsized after being created. The firm is completely safe after 40 periods.

5 Conclusions

We have proposed a dynamic–contracting model that can incorporate the possibility of recapitalization and have shown how should a firm choose between two options, liquidation or recapitalization, when facing financial difficulties. Our model analyzes the relationship between an entrepreneur and investors regarding the financing of a business project whose cash flows are determined by the unobserved effort of the former. To provide incentives to the entrepreneur, the investors can either impose a pecuniary penalty or terminate the project, which corresponds to recapitalization or liquidation of the firm, respectively. These choices have been determined via the characterization of the optimal contract. Which mechanism should be employed is based on a costs comparison.

[14] The plot corresponds to the choice $\tau = 0.15$.



(a) (b)

Figure 9: Dynamics of the entrepreneur's continuation utility in the case of high recapitalization costs

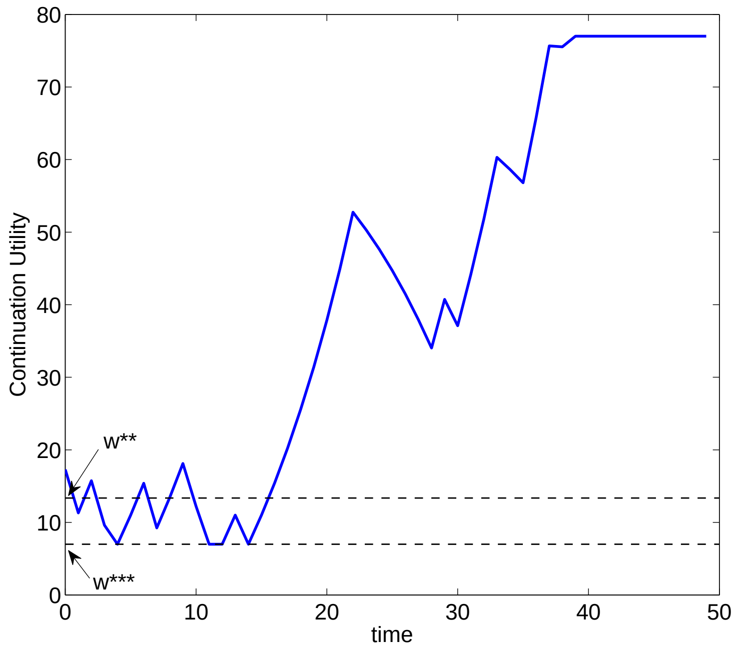


Figure 10: Dynamics of the entrepreneur's continuation utility in the case of low recapitalization costs

A Proofs

Proof of Proposition 1. We first prove that there exists a unique, continuous and bounded solution V to the Maximization Problem (7) subject to the Constraints (3) - (6). Let $\mathcal{C}_b(\mathbb{R}_+)$ be the space of bounded, continuous, real–valued functions defined on $\mathbb{R}_+$ and define an operator T on $\mathcal{C}_b(\mathbb{R}_+)$ via the program:

$$Tv(w) = \max_{x,\overline{c},\underline{c},\overline{w},\underline{w}} \Big\{ x \Big[\overline{R} + p\left(U(\overline{c}) - \overline{c}\right) + (1-p)\left(U(\underline{c}) - \underline{c}\right) + \frac{pv(\overline{w}) + (1-p)v(\underline{w})}{1+r} \Big] + (1-x)L \Big\} \tag{15}$$

for all non–negative w, subject to

$$x\Big[pU(\overline{c}) + (1-p)U(\underline{c}) + \frac{p\overline{w} + (1-p)\underline{w}}{1+r}\Big] = w; \tag{16}$$

$$U(\overline{c}) - U(\underline{c}) + \frac{\overline{w} - \underline{w}}{1+r} \geq \frac{B}{\Delta p};$$
$$U(\underline{c}) + \frac{\underline{w}}{1+r} \geq 0;$$
$$(x, \overline{w}, \underline{w}) \in [0,1] \times \mathbb{R}_+^2.$$

Proving there exists an unique, continuous and bounded function V that solves the Maximization Problem (7) subject to the Constraints (3) - (6) amounts to showing that T has a unique fixed point. To this end, we show below that T maps $C_b(R_+)$ into itself and that it is a contraction, and then invoke Banach's Fixed Point Theorem.

Consider v in $\mathcal{C}_b(\mathbb{R}_+)$. First observe that $U(c) - c$ is smaller than or equal to zero for all c in $\mathbb{R}$, which together with the fact that the map $w \mapsto v(w)$ is bounded implies that Tv is bounded above. Let $\hat{w}$ be the smallest point at which the mapping $w \mapsto v(w)$ attains its maximum[15]. It is clear that

$$\left(x, \overline{c}, \underline{c}, \overline{w}, \underline{w}\right) = \Big(1, \frac{1}{p}(w - \frac{\hat{w}}{1+r}), 0, \hat{w}, \hat{w}\Big)$$

is a solution to the program for all w greater than $\frac{pB}{\Delta p} + \frac{\hat{w}}{1+r}$. This yields

$$Tv(w) = Tv\Big(\frac{pB}{\Delta p} + \frac{\hat{w}}{1+r}\Big) = \overline{R} + \frac{v(\hat{w})}{1+r} \quad \text{for all } w \in \Big(\frac{pB}{\Delta p} + \frac{\hat{w}}{1+r}, \infty\Big).$$

For any value of w on $\big[0, \frac{pB}{\Delta p} + \frac{\hat{w}}{1+r}\big]$, there is no loss of generality to restrict $(x, \overline{c}, \underline{c}, \overline{w}, \underline{w})$ to

$$D := [0,1] \times \left[-\frac{\hat{w}}{(1+\tau)(1+r)}, \frac{B}{\Delta p}\right] \times \left[-\frac{\hat{w}}{(1+\tau)(1+r)}, 0\right] \times [0, \hat{w}]^2.$$

It then follows that $Tv(w)$ is also bounded below and that the mapping $w \mapsto Tv(w)$ is continuous on $\left[0, \frac{pB}{\Delta p} + \frac{\hat{w}}{1+r}\right]$, since the function under the maximization operator is continuous and the set D is compact. Given that we have continuous pasting at $w = pB/\Delta p + \hat{w}/(1+r)$, we conclude that Tv belongs to $\mathcal{C}_b(\mathbb{R}_+)$, i.e. the operator T maps $C_b(R_+)$ into itself. It is straightforward to verify that T satisfies Blackwell's sufficient conditions to be a contraction: monotonicity and discounting. Hence, we may use Banach's Fixed Point Theorem to conclude that T has an unique fixed point $V \in C_b(R_+)$.

Next we show that V is non decreasing. For this purpose, let w and w' be two non–negative values such that w' is greater than w. Let $(x, \overline{c}, \underline{c}, \overline{w}, \underline{w})$ be a solution to the program that defines $Tv(w)$. Observe there is no loss of generality in assuming that x is strictly positive, since otherwise the project would be downsized to zero. Consider $\overline{c}'$ such that

$$p\big(U(\overline{c}') - U(\overline{c})\big) = \frac{w' - w}{x} > 0.$$

[15] Since $v(w)$ is a continuous, bounded function, such a $\hat{w}$ exists.

$U(c)$ is an increasing function, thus $\overline{c}'$ is greater than $\overline{c}$. We claim that the choice $\big(x, \overline{c}', \underline{c}, \overline{w}, \underline{w}\big)$ is feasible for the program that defines $Tv(w')$. Indeed

$$x\left(pU(\overline{c}') + (1-p)U(\underline{c}) + \frac{p\overline{w} + (1-p)\underline{w}}{1+r}\right) = x\left(\frac{w'-w}{x} + \frac{w}{x}\right) = w',$$

and

$$U(\overline{c}') - U(\underline{c}) + \frac{\overline{w} - \underline{w}}{1+r} > U(\overline{c}) - U(\underline{c}) + \frac{\overline{w} - \underline{w}}{1+r} \geq \frac{B}{\Delta p}.$$

Due to the fact that the mapping $c \mapsto U(c) - c$ is non decreasing, we have

$$\begin{aligned} Tv(w') &\geq x\Big[\overline{R} + p\big(U(\overline{c}') - \overline{c}'\big) + (1-p)\big(U(\underline{c}) - \underline{c}\big) + \frac{pV(\overline{w}) + (1-p)V(\underline{w})}{1+r}\Big] + (1-x)L \\ &\geq x\Big[\overline{R} + p\big(U(\overline{c}) - \overline{c}\big) + (1-p)\big(U(\underline{c}) - \underline{c}\big) + \frac{pV(\overline{w}) + (1-p)V(\underline{w})}{1+r}\Big] + (1-x)L \\ &= V(w), \end{aligned}$$

i.e. Tv is a non–decreasing function.

In order to show that V is concave, we define, for all w greater than or equal to $pB/\Delta p$, the following auxiliary program:

$$T^c v(w) := \max_{\overline{c}, \underline{c}, \overline{w}, \underline{w}} \left\{\overline{R} + p\left(U(\overline{c}) - \overline{c}\right) + (1-p)\left(U(\underline{c}) - \underline{c}\right) + \frac{pv(\overline{w}) + (1-p)v(\underline{w})}{1+r}\right\}$$

subject to

$$\begin{aligned} &pU(\overline{c}) + (1-p)U(\underline{c}) + \frac{p\overline{w} + (1-p)\underline{w}}{1+r} = w, \quad \big(\overline{w}, \underline{w}\big) \in \mathbb{R}^2_+, \\ &U(\overline{c}) - U(\underline{c}) + \frac{\overline{w} - \underline{w}}{1+r} \geq \frac{B}{\Delta p} \quad \text{and} \quad U(\underline{c}) + \frac{\underline{w}}{1+r} \geq 0. \end{aligned}$$

We may then write

$$Tv(w) = \max_{x, w^c} \big\{xT^c v(w^c) + (1-x)L\big\} \quad \text{for all } w \geq 0$$

subject to

$$xw^c = w,\ x \in [0,1] \text{ and } w^c \in \Big[\frac{pB}{\Delta p}, \infty\Big).$$

We first show that if v is concave so is $T^c v$. To this end, consider w and w' greater than or equal to $pB/\Delta p$ and let $\big(\overline{c}, \underline{c}, \overline{w}, \underline{w}\big)$ and $\big(\overline{c}', \underline{c}', \overline{w}', \underline{w}'\big)$ be solutions to the programs that define $T^c v(w)$ and $T^c v(w')$, respectively[16]. For λ in $(0,1)$ define $w_\lambda := \lambda w + (1-\lambda)w'$, $\overline{w_\lambda} := \lambda \overline{w} + (1-\lambda)\overline{w}'$, $\underline{w_\lambda} := \lambda \underline{w} + (1-\lambda)\underline{w}'$, $\overline{c_\lambda} := U^{-1}\big(\lambda U(\overline{c}) + (1-\lambda)U(\overline{c}')\big)$ and $\underline{c_\lambda} := U^{-1}\big(\lambda U(\underline{c}) + (1-\lambda)U(\underline{c}')\big)$. Observe that

$$U(\underline{c_\lambda}) + \frac{\underline{w_\lambda}}{1+r} \geq 0 \quad \text{and} \quad U(\overline{c_\lambda}) - U(\underline{c_\lambda}) + \frac{\overline{w_\lambda} - \underline{w_\lambda}}{1+r} \geq \frac{B}{\Delta p}.$$

Hence, $\big(\overline{c_\lambda}, \underline{c_\lambda}, \overline{w_\lambda}, \underline{w_\lambda}\big)$ is feasible for the program that defines $T^c v(w_\lambda)$, which implies

$$T^c v(w_\lambda) \geq \overline{R} + p\left(U(\overline{c_\lambda}) - \overline{c_\lambda}\right) + (1-p)\left(U(\underline{c_\lambda}) - \underline{c_\lambda}\right) + \frac{pv(\overline{w_\lambda}) + (1-p)v(\underline{w_\lambda})}{1+r}$$

[16] Given the relation between Tv and $T^c v$ and the fact that for any $w \geq 0$ there exist a solution to the program that defines $Tv(w)$, the same holds true in the case of $T^c v(w)$ for all $w \geq \frac{pB}{\Delta p}$.

By concavity of v we have $v(\overline{w_\lambda}) \geq \lambda v(\overline{w}) + (1-\lambda)v(\overline{w}')$ and $v(\underline{w_\lambda}) \geq \lambda v(\underline{w}) + (1-\lambda)v(\underline{w}')$. Given that $U(.)$ is concave and non decreasing, we have $\overline{c_\lambda} \leq \lambda\overline{c} + (1-\lambda)\overline{c}'$ so

$$U(\overline{c_\lambda}) - \overline{c_\lambda} \geq \lambda\left[U(\overline{c}) - \overline{c}\right] + (1-\lambda)\left[U(\overline{c}') - \overline{c}'\right].$$

The analogous relation holds for $U(\underline{c_\lambda}) - \underline{c_\lambda}$, therefore

$$T^c v(w_\lambda) \geq \lambda T^c v(w) + (1-\lambda)T^c v(w')$$

as required.

Next we establish the concavity of Tv. We have

$$Tv(w) = \max_{w^c}\Big\{w\Big[\frac{T^c v(w^c) - L}{w^c}\Big]\Big\} + L$$

for all non–negative w, subject to

$$w^c \geq \max\Big\{w, \frac{pB}{\Delta p}\Big\}.$$

Define $\hat{w}^* := \hat{w}/(1+r) + pB/\Delta p$. In analogous fashion as we did for Tv, one can verify that $T^c v$ is continuous on its domain and constant over $\big[\hat{w}^*, \infty\big)$. The latter implies that the mapping $w^c \mapsto (T^c v(w^c) - L)/w^c$ reaches its maximum on $\big[\frac{pB}{\Delta p}, \hat{w}^*\big]$. Denote the $\arg\max\big\{(T^c v(w^c) - L)/w^c\big\}$ by $\big[\underline{w}^c, \overline{w}^c\big]$, with $\underline{w}^c$ greater than or equal to $pB/\Delta p$. We then have

$$\operatorname*{arg\,max}_{w^c \geq \max\left\{w, \frac{pB}{\Delta p}\right\}} \frac{T^c v(w^c) - L}{w^c} = \begin{cases} w \text{ if } w \geq \overline{w}^c; \\ [w, \overline{w}^c] \text{ if } \overline{w}^c > w \geq \underline{w}^c; \\ [\underline{w}^c, \overline{w}^c] \text{ if } \underline{w}^c > w \geq 0. \end{cases}$$

This implies that

$$Tv(w) = \begin{cases} T^c v(w) \text{ if } w \geq \overline{w}^c; \\ w\,K + L \text{ if } 0 \leq w < \overline{w}^c. \end{cases}$$

Here K equals $(T^c v(\underline{w}^c) - L)/\underline{w}^c$, which is independent of w; therefore, Tv is concave.

Now, we show that V is strictly increasing on $[0, \frac{1+r}{r}\frac{pB}{\Delta p})$ and constant on $[\frac{1+r}{r}\frac{pB}{\Delta p}, +\infty)$. Indeed, we have already shown that

$$V(w) = \overline{R} + \frac{V(\hat{w})}{1+r} \text{ if } w \geq \frac{pB}{\Delta p} + \frac{\hat{w}}{1+r},$$

where $\hat{w}$ is smallest point at which the mapping $w \mapsto V(w)$ attains its maximum. If V were constant on some interval $[w_1, w_2]$ with w_2 smaller than $pB/\Delta p + \hat{w}/(1+r)$, then by concavity it would be constant on $\big[w_1, \frac{pB}{\Delta p} + \frac{\hat{w}}{1+r}\big]$. This would imply that $\big(x, \overline{c}, \underline{c}, \overline{w}, \underline{w}\big) = \big(1, \frac{B}{\Delta p}, 0, \hat{w}, \hat{w}\big)$ would be a solution to the program that defines $V(w_1)$ and so, $w_1 = \frac{pB}{\Delta p} + \frac{\hat{w}}{1+r}$, which is a contradiction. Hence, $V(w)$ is strictly increasing over $\big[0, \frac{pB}{\Delta p} + \frac{\hat{w}}{1+r}\big)$. Moreover, it is constant over $\big[\frac{pB}{\Delta p} + \frac{\hat{w}}{1+r}, +\infty\big)$, which implies

$$\frac{pB}{\Delta p} + \frac{\hat{w}}{1+r} = \min\Big\{\operatorname{arg\,max}\big\{V(w)\big\}\Big\} \Rightarrow \frac{pB}{\Delta p} + \frac{\hat{w}}{1+r} = \hat{w} \Rightarrow \hat{w} = \frac{1+r}{r}\frac{pB}{\Delta p}.$$

Finally, from the relation $V(\hat{w}) = \overline{R} + V(\hat{w})/(1+r)$ we obtain $V(w) = \overline{R}(1+r)/r$ for all w

that are greater than or equal to $\frac{(1+r)pB}{r\Delta p}$.

Q.E.D.

Proof of Lemma 1. Let us first assume that, for w strictly smaller than $\frac{1+r}{r}\frac{pB}{\Delta p}$, there exists a solution $\big(x, \overline{c}, \underline{c}, \overline{w}, \underline{w}\big)$ to the Maximization Problem (7) subject to the Constraints (3) - (6) such that $\underline{c} > 0$. Since $\underline{c}$ is strictly greater than zero, there exists a $\varepsilon > 0$ such that $\underline{c} - \varepsilon \geq 0$. Define

$$\underline{c}' := \underline{c} - \varepsilon \quad \text{and} \quad \underline{w}' := \underline{w} + (1+r)\varepsilon > \underline{w}.$$

It is straightforward to verify that $\big(x, \overline{c}, \underline{c}', \overline{w}, \underline{w}'\big)$ is also feasible for the Maximization Problem (7). Since $V(w)$ is a non–decreasing function and $U(c) - c$ equals zero for all non–negative c, we have:

$$x\Big[\overline{R} + p\big(U(\overline{c}) - \overline{c}\big) + (1-p)\big(U(\underline{c}) - \underline{c}\big) + \frac{pV(\overline{w}) + (1-p)V(\underline{w})}{1+r}\Big] + (1-x)L$$

$$\leq x\Big[\overline{R} + p\big(U(\overline{c}) - \overline{c}\big) + (1-p)\big(U(\underline{c}') - \underline{c}'\big) + \frac{pV(\overline{w}) + (1-p)V(\underline{w}')}{1+r}\Big] + (1-x)L.$$

Therefore, there is no loss of generality in restricting the set of feasible solutions to the Maximization Problem (7) by assuming that $\underline{c}$ is non–positive whenever w is strictly smaller than $\frac{1+r}{r}\frac{pB}{\Delta p}$. By analogous argument we may also assume that, on the same range for w, $\overline{c}$ is non–positive.

Q.E.D.

Proof of Lemma 2. Constraints (3) - (5) imply x is smaller than or equal to $w/\big(\frac{pB}{\Delta p}\big)$. Given that x is bounded above by one we have

$$0 \leq x \leq \min\Big\{1, \frac{w}{\frac{pB}{\Delta p}}\Big\}.$$

We first show, by the way of contradiction, that for w on $\big[\frac{pB}{\Delta p}, \frac{1+r}{r}\frac{pB}{\Delta p}\big)$ it is optimal to choose x equal to 1[17]. To this end, let us assume that, for w on the said interval, there exists a solution $\big(x_1, \overline{c}_1, \underline{c}_1, \overline{w}_1, \underline{w}_1\big)$ to the Maximization Problem (9) such that x_1 is strictly smaller than 1. We have[18]:

$$\overline{R} + \frac{p}{1+r}\Big(V(\overline{w}_1) - \frac{\tau}{1+\tau}\overline{w}_1\Big) + \frac{1-p}{1+r}\Big(V(\underline{w}_1) - \frac{\tau}{1+\tau}\underline{w}_1\Big) > L \tag{17}$$

Let us define the non–negative quantities

$$\alpha := \frac{w}{x_1} - w, \; \beta := U(\overline{c}_1) - U(\underline{c}_1) + \frac{\overline{w}_1 - \underline{w}_1}{1+r} - \frac{B}{\Delta p}, \text{ and } \gamma := U(\underline{c}_1) + \frac{\underline{w}_1}{1+r}$$

(in fact α is strictly positive). From Constraint (3) and the fact that w is greater than or equal

[17]We have already established, as a consequence of Theorem 1, that $x \equiv 1$ for w greater than $\frac{1+r}{r}\frac{pB}{\Delta p}$.

[18]Otherwise, we have

$$x_1\Big[\overline{R} + \frac{p}{1+r}\Big(V(\overline{w}_1) - \frac{\tau}{1+\tau}\overline{w}_1\Big) + \frac{1-p}{1+r}\Big(V(\underline{w}_1) - \frac{\tau}{1+\tau}\underline{w}_1\Big)\Big] + (1-x_1)L \leq L$$

which contradicts the fact that $\big(x_1, \overline{c}_1, \underline{c}_1, \overline{w}_1, \underline{w}_1\big)$ is the solution to the Maximization Problem.

to $\frac{pB}{\Delta p}$ we have

$$p\beta + \frac{pB}{\Delta p} + \gamma = w + \alpha \geq \frac{pB}{\Delta p} + \alpha.$$

Therefore

$$p\beta + \gamma \geq \alpha. \tag{18}$$

Next we define $\overline{c}_2$ and $\underline{c}_2$ via the relations $U(\overline{c}_2) = U(\overline{c}_1) - \varepsilon$, and $U(\underline{c}_2) = U(\underline{c}_1) - \omega$, where ε and ω are solutions to the system

$$\begin{gathered} p\varepsilon + (1-p)\omega = \alpha, \\ \varepsilon, \omega \geq 0, \\ \max\big\{0, \alpha - p\beta\big\} \leq \omega \leq \min\Big\{\gamma, \frac{\alpha}{1-p}\Big\}. \end{gathered}$$

This system has a solution, since, by Inequality (18), each of the elements under the $\max\{\cdot,\cdot\}$ operator is bounded above by each of the elements under the $\min\{\cdot,\cdot\}$ one. Let $x_2 = 1$, $\overline{w}_2 = \overline{w}_1$, and $\underline{w}_2 = \underline{w}_1$. It is straightforward to verify that the quintuple $\big(x_2, \overline{c}_2, \underline{c}_2, \overline{w}_2, \underline{w}_2\big)$ satisfies the Constraints (3) - (6) and (10). Therefore $\big(x_2, \overline{c_2}, \underline{c_2}, \overline{w_2}, \underline{w_2}\big)$ are feasible for the Maximization Problem (9). Furthermore, we have

$$\begin{aligned} V(w) &= \frac{\tau}{1+\tau}w + x_1\Big[\overline{R} + \frac{p}{1+r}\Big(V(\overline{w}_1) - \frac{\tau}{1+\tau}\overline{w}_1\Big) \\ &\quad + \frac{1-p}{1+r}\Big(V(\underline{w}_1) - \frac{\tau}{1+\tau}\underline{w}_1\Big)\Big] + (1-x_1)L \\ &< \frac{\tau}{1+\tau}w + \overline{R} + \frac{p}{1+r}\Big(V(\overline{w}_1) - \frac{\tau}{1+\tau}\overline{w}_1\Big) + \frac{1-p}{1+r}\Big(V(\underline{w}_1) - \frac{\tau}{1+\tau}\underline{w}_1\Big) \\ &= \frac{\tau}{1+\tau}w + x_2\Big[\overline{R} + \frac{p}{1+r}\Big(V(\overline{w}_2) - \frac{\tau}{1+\tau}\overline{w}_2\Big) + \frac{1-p}{1+r}\Big(V(\underline{w}_2) - \frac{\tau}{1+\tau}\underline{w}_2\Big)\Big], \end{aligned}$$

where the inequality is due to x_1 being smaller than 1 and Inequality (17). This contradicts the optimality of $\big(x_1, \overline{c}_1, \underline{c}_1, \overline{w}_1, \underline{w}_1\big)$, thus, when w belongs to $\big[\frac{pB}{\Delta p}, \frac{1+r}{r}\frac{pB}{\Delta p}\big)$ it is optimal to let $x = 1$.

Next we look at the case where w belongs to $\big[0, \frac{pB}{\Delta p}\big)$ and show that it is optimal to choose x equal to $w/\big(\frac{pB}{\Delta p}\big)$. In analogous fashion to the previous part, assume that, for w on the said interval, there exists a solution $\big(x_3, \overline{c}_3, \underline{c}_3, \overline{w}_3, \underline{w}_3\big)$ to the Maximization Problem (9) such that x_3 is strictly smaller than $w/\big(\frac{pB}{\Delta p}\big)$. Notice that $\underline{c}_3$ and $\overline{c}_3$ satisfy the non–positivity Constraint (10). Let us define the non–negative quantities

$$\alpha_1; = U(\overline{c_3}) - U(\underline{c_3}) + \frac{\overline{w_3} - \underline{w_3}}{1+r} - \frac{B}{\Delta p}, \beta_1 = U(\underline{c_3}) + \frac{\underline{w_3}}{1+r} \text{ and } \gamma_1 = \frac{w}{x_3} - \frac{pB}{\Delta p}$$

(in fact γ_1 is strictly positive). Next define $\overline{c}_4$ and $\underline{c}_4$ through the relations $U(\overline{c}_4) = U(\overline{c}_3) - \alpha_1 - \beta_1$ and $U(\underline{c}_4) = U(\underline{c}_3) - \beta_1$ and set $x_4 = w/\big(\frac{pB}{\Delta p}\big)$, $\overline{w}_4 = \overline{w}_3$ and $\underline{w}_4 = \underline{w}_3$. It is immediate to check that $(x_4, \overline{c}_4, \underline{c}_4, \overline{w}_4, \underline{w}_4)$ satisfies the Constraints (3) - (6) and (10). Moreover:

$$\begin{aligned}
V(w) &= \frac{\tau}{1+\tau}w + (1-x_3)L \\
&+ x_3\Big[\overline{R} + \frac{p}{1+r}\Big(V(\overline{w}_3) - \frac{\tau}{1+\tau}\overline{w}_3\Big) + \frac{1-p}{1+r}\Big(V(\underline{w}_3) - \frac{\tau}{1+\tau}\underline{w}_3\Big)\Big] \\
&< \frac{\tau}{1+\tau}w + (1-x_4)L \\
&+ x_4\Big[\overline{R} + \frac{p}{1+r}\Big(V(\overline{w}_4) - \frac{\tau}{1+\tau}\overline{w}_4\Big) + \frac{1-p}{1+r}\Big(V(\underline{w}_4) - \frac{\tau}{1+\tau}\underline{w}_4\Big)\Big],
\end{aligned}$$

which contradicts the optimality of $(x_3, \overline{c}_3, \underline{c}_3, \overline{w}_3, \underline{w}_3)$.

Q.E.D.

Proof of Proposition 2. We have to find the optimal continuation utilities within the following two cases:

High recapitalization costs $\left(\mathbf{V'(0)} < \frac{\tau}{\mathbf{1+\tau}}\right)$**.** We know from Proposition 1 that the value function V is concave. Therefore, the high–recapitalization–costs condition $V_l'(0) < \tau/(1+\tau)$ implies that $V_l'(w)$ is smaller than $\tau/(1+\tau)$ for all non–negative w. In other words, the mapping $w \mapsto V_l(w) - w\tau/(1+\tau)$ is decreasing on $\mathbb{R}_+$, thus the term

$$\overline{R} + \frac{p}{1+r}\big(V_l(\overline{w}) - \frac{\tau}{1+\tau}\overline{w}\big) + \frac{1-p}{1+r}\big(V_l(\underline{w}) - \frac{\tau}{1+\tau}\underline{w}\big)$$

in the Objective Function (9) is maximized when the (lower–bound) constraints on $\overline{w}$ and $\underline{w}$ are tight. Consequently, for w in $\left[0, \frac{1+r}{r}\frac{pB}{\Delta p}\right)$ we have, at the optimum

$$\overline{w} = (1+r)\Big(w^c + \frac{(1-p)B}{\Delta p}\Big) \quad \text{and} \quad \underline{w} = (1+r)\Big(w^c - \frac{pB}{\Delta p}\Big). \tag{19}$$

Substituting Expression (19) into Expression (12) we obtain that both $\overline{c}$ and $\underline{c}$ equal 0.

Low recapitalization costs $\left(\mathbf{V'(0)} > \frac{\tau}{\mathbf{1+\tau}}\right)$**.** When $V'(0)$ is greater than $\tau/(1+\tau)$, there exists a positive $\tilde{w}$ such that the mapping $w \mapsto V(w) - w\tau/(1+\tau)$ is increasing below $\tilde{w}$ and decreasing above it. We first pin down $\tilde{w}$ in the following auxiliary lemma:

Lemma 3. *If $V_r'(0)$ is greater than $\tau/(1+\tau)$, then the mapping*

$$w \mapsto V_r(w) - \frac{\tau}{1+\tau}w$$

is maximized at $\tilde{w} = pB/\Delta p$.

Proof. Define the function $H : \left[0, \frac{1+r}{r}\frac{pB}{\Delta p}\right) \to \mathbb{R}_+$ via

$$H(w) := V(w) - \frac{\tau}{1+\tau}w.$$

We have from the Maximization Program (9) that $H(w)$ is determined by the following program:

$$H(w) = \max_{x,\overline{w},\underline{w}} \Big\{x\Big[\overline{R} + \frac{pH(\overline{w}) + (1-p)H(\underline{w})}{1+r}\Big] + (1-x)L\Big\}$$

subject to

$$\frac{\overline{w}}{1+r} \geq \frac{w}{x} + \frac{(1-p)B}{\Delta p}, \quad \frac{\underline{w}}{1+r} \geq \frac{w}{x} - \frac{pB}{\Delta p} \tag{20}$$

and

$$x = \min\Big\{1, w/\big(\frac{pB}{\Delta p}\big)\Big\}, \tag{21}$$

where the Constraints (20) correspond to Constraints (13) with w/x substituted for w^c. Since V is concave so is H and clearly $H(0) = L$. Moreover, if $V'(0)$ is strictly greater than $\tau/(1+\tau)$, then $H'(0)$ is strictly greater than zero.

We first show that the mapping $w \mapsto H(w)$ is non–increasing whenever w is greater than or equal to $pB/\Delta p$. To this end consider w_a and w_b such that

$$w_a \geq w_b \geq \frac{pB}{\Delta p}.$$

Let $\big(1, \overline{w}_a, \underline{w}_a\big)$ be a solution to the program that defines $H\big(w_a\big)$. Hence, from Constraints (20) we have

$$\frac{\overline{w}_a}{1+r} \geq w_a + \frac{(1-p)B}{\Delta p} \geq w_b + \frac{(1-p)B}{\Delta p} \quad \text{and} \quad \frac{\underline{w}_a}{1+r} \geq w_a - \frac{pB}{\Delta p} \geq w_b - \frac{pB}{\Delta p},$$

which means $\big(1, \overline{w}_a, \underline{w}_a\big)$ is feasible for the program that defines $H(w_b)$. Then we get

$$H(w_b) \geq \overline{R} + \frac{pH(\overline{w}_a) + (1-p)H(\underline{w}_a)}{1+r} = H(w_a),$$

as required.

Next we show that the mapping $w \mapsto H(w)$ is non–decreasing when w belongs to $\big[0, \frac{pB}{\Delta p}\big)$. Notice that in such case, it holds that x equals $w/\big(\frac{pB}{\Delta p}\big)$. Thus, on the interval at hand the program defining the function $H(\cdot)$ can be rewritten as follows:

$$H(w) = \max_{\overline{w}, \underline{w}} \ \frac{w}{\frac{pB}{\Delta p}} \left[\overline{R} + \frac{pH(\overline{w}) + (1-p)H(\underline{w})}{1+r}\right] + \left(1 - \frac{w}{\frac{pB}{\Delta p}}\right) L$$

subject to

$$\frac{\overline{w}}{1+r} \geq \frac{B}{\Delta p}, \quad \text{and} \quad \frac{\underline{w}}{1+r} \geq 0.$$

First, we prove that $H(w)$ is greater than or equal to L for all w on $\big[0, \frac{pB}{\Delta p}\big)$. By hypothesis $H'(0)$ is strictly positive, thus there exists ε greater than 0 such that $H(\cdot)$ is an increasing function on $(0, \varepsilon]$. Since $H(0)$ equals L, this implies that $H(w_c)$ is greater than or equal to L for any w_c on $(0, \varepsilon]$. Let $\big(\overline{w}_c, \underline{w}_c\big)$ be a solution to the program that defines $H\big(w_c\big)$, then

$$H(w_c) = \ \frac{w_c}{\frac{pB}{\Delta p}} \left[\overline{R} + \frac{pH(\overline{w}_c) + (1-p)H(\underline{w}_c)}{1+r}\right] + \left(1 - \frac{w_c}{\frac{pB}{\Delta p}}\right) L \geq L;$$

therefore

$$\overline{R} + \frac{pH(\overline{w}_c) + (1-p)H(\underline{w}_c)}{1+r} - L \geq 0. \tag{22}$$

Clearly, for any w_d on $\big(\varepsilon, \frac{pB}{\Delta p}\big)$ the vector $\big(\overline{w}_c, \underline{w}_c\big)$ is a feasible choice for the program that defines

$H(w_d)$. This implies

$$\begin{aligned} H(w_d) &\geq \frac{w_d}{\frac{pB}{\Delta p}} \left[\overline{R} + \frac{pH(\overline{w}_c) + (1-p)H(\underline{w}_c)}{1+r} \right] + \left(1 - \frac{w_d}{\frac{pB}{\Delta p}} \right) L \\ &\geq \frac{w_c}{\frac{pB}{\Delta p}} \left[\overline{R} + \frac{pH(\overline{w}_c) + (1-p)H(\underline{w}_c)}{1+r} \right] + \left(1 - \frac{w_c}{\frac{pB}{\Delta p}} \right) L = H(w_c) \geq L, \end{aligned}$$

where the second inequality follows from Inequality (22). We conclude that $H(w)$ is greater than or equal to L for all w on $\left[0, \frac{pB}{\Delta p}\right)$.

Now let w_e and w_f be such that:

$$0 \leq w_f \leq w_e < \frac{pB}{\Delta p}.$$

Hence, $H(w_f)$ is greater than or equal to L. Let $(\overline{w}_f, \underline{w}_f)$ be a solution to the program that defines $H(w_f)$. We thus have:

$$H(w_f) = \frac{w_f}{\frac{pB}{\Delta p}} \left[\overline{R} + \frac{pH(\overline{w}_f) + (1-p)H(\underline{w}_f)}{1+r} \right] + \left(1 - \frac{w_f}{\frac{pB}{\Delta p}} \right) L \geq L$$

which implies that

$$\overline{R} + \frac{pH(\overline{w}_f) + (1-p)H(\underline{w}_f)}{1+r} - L \geq 0. \tag{23}$$

Since $(\overline{w}_f, \underline{w}_f)$ is feasible for the program that defines $H(w_e)$, we get

$$\begin{aligned} H(w_e) &\geq \frac{w_e}{\frac{pB}{\Delta p}} \left(\overline{R} + \frac{pH(\overline{w}_f) + (1-p)H(\underline{w}_f)}{1+r} \right) + \left(1 - \frac{w_e}{\frac{pB}{\Delta p}} \right) L \\ &\geq \frac{w_f}{\frac{pB}{\Delta p}} \left(\overline{R} + \frac{pH(\overline{w}_f) + (1-p)H(\underline{w}_f)}{1+r} \right) + \left(1 - \frac{w_f}{\frac{pB}{\Delta p}} \right) L = H(w_f), \end{aligned}$$

where the second inequality follows from Inequality (23). This concludes the proof of the auxiliary lemma.

□

With Lemma 3 in hand, we now continue with the proof of Proposition 2. Since w^c is greater than or equal to $pB/\Delta p$, the right–hand side of the first expression in Constraints (13) is strictly greater than $pB/\Delta p$. Therefore, a $\overline{w}$ satisfying (13) must be larger than $pB/\Delta p$. As a consequence, at the optimum, we have

$$\overline{w} = (1+r) \left(w^c + \frac{(1-p)B}{\Delta p} \right),$$

which in turn implies $\overline{c} = 0$. Relative to $\underline{w}$, we follow an analogous reasoning and we get, using the Thresholds (14), the following result:

$$\underline{w} = \begin{cases} (1+r) \left(w^c - \frac{pB}{\Delta p} \right) & if \quad w^{**} \leq w^c < w^*; \\ \frac{pB}{\Delta p} & if \quad w^{***} \leq w^c < w^{**}. \end{cases}$$

The optimal choice for $\underline{c}$ is derived from Expression (12). We find that if w^c belongs to $\big[w^{***}, w^{**}\big)$ then $\underline{c}$ takes a negative value.

Q.E.D.

Proof of Proposition 3. We need to establish that the entrepreneur's utility in the implementation is the same as in the optimal contract.

- When $m_t \geq m^*$, which is equivalent to $w_t^c \geq w^*$, the optimal contract requires that:

$$\overline{c}_t = w^c - w^* + \frac{B}{\Delta p}, \qquad \underline{c}_t = w^c - w^* \quad \text{and} \quad \overline{w}_{t+1} = \underline{w}_{t+1} = w^*.$$

  In the implementation, we specify that the firm distributes a dividend

$$d_t = m_t - m^* + \frac{1}{k}\frac{B}{\Delta p}\mathbb{1}_{\{R_t = R_+\}},$$

  where $\mathbb{1}_{\{\cdot\}}$ is the zero–one indicator function. Given that the entrepreneur holds a fraction k of the equity, she will receive an amount of cash kd_t, which is equal to c_t. The amount of size–adjusted cash reserves in the next period is as follows:

$$m_{t+1} = (1+r)(m_t + R_t - \overline{R} - d_t) = m^*,$$

  which implies that $w_{t+1} = km_{t+1} = w^*$, as required.

- When m_t belongs to $[m^{**}, m^*)$, the optimal contract specifies that $\overline{c}_t = \underline{c}_t = 0$ and

$$\overline{w}_{t+1} = (1+r)(w_t^c + \frac{(1-p)B}{\Delta p}) \quad \text{and} \quad \underline{w}_{t+1} = (1+r)(w_t^c - \frac{B}{\Delta p}).$$

  In the implementation, the firm distributes no dividends, which implies that the entrepreneur receives no cash compensation, as required. Since the firm pays $\overline{R}$ to its debt holders, the size–adjusted cash reserves evolve according to:

$$m_{t+1} = (1+r)(m_t + R_t - \overline{R}) = \begin{cases} (1+r)\big(m_t + (1-p)(R_+ - R_-)\big) & \text{if } R_t = R_+; \\ (1+r)\big(m_t - p(R_+ - R_-)\big) & \text{if } R_t = R_-. \end{cases}$$

  It is immediate to check that

$$w_{t+1} = k\, m_{t+1} = \begin{cases} (1+r)\big(w_t^c + (1-p)B/\Delta p\big) & \text{if } R_t = R_+; \\ (1+r)\big(w_t^c - pB/\Delta p\big) & \text{if } R_t = R_-, \end{cases}$$

  as required by the optimal contract.

- When $m_t < m^{**}$, in the case of low recapitalization costs, the optimal contract specifies that:

$$\overline{c}_t = 0, \quad \text{and} \quad \underline{c}_t = -\frac{1}{1+\tau}(w^{**} - w_t^c)$$

  and

$$\overline{w}_{t+1} = (1+r)(w_t^c + \frac{(1-p)B}{\Delta p}) \quad \text{and} \quad \underline{w}_{t+1} = \frac{pB}{\Delta p}.$$

  In the implementation, following a high cash flow, the firm is in its normal functioning state: it pays the promised repayment to its debt holders and does not distribute dividends.

Hence, the entrepreneur receives no cash, as required. The evolution of the size–adjusted cash reserves is as follows:

$$m_{t+1} = (1+r)(m_t + R_+ - \overline{R}),$$

therefore $\overline{w}_{t+1} = km_{t+1} = (1+r)(w_t^c + \frac{(1-p)B}{\Delta p})$. However, following a low cash flow, the firm is restructured: an amount of $i_t := m^{**} - m_t$ of new equity is issued. Since the entrepreneur takes up a fraction k of the new issuance, the total costs for her are equal to $ki_t = w^{**} - w_t^c$, which means that $\underline{c}_t = -\frac{1}{1+\tau}(w^{**} - w_t^c)$, as required. Moreover, since the current debt service is written off by τi_t, the size–adjusted cash reserves evolve as follows:

$$m_{t+1} = (1+r)(m_t + R_- - (\overline{R} - \tau i_t) + \frac{1}{1+\tau} i_t).$$

It is immediate to check that $\underline{w}_{t+1} = km_{t+1} = \frac{pB}{\Delta p}$.

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